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On the Chern-Simons Topological Term at Finite Temperature

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Abstract

The parity-violating topological term in the effective action for 2+1 massive fermions is computed at finite temperature in the presence of a constant background field strength tensor. Gauge invariance of the finite-temperature effective action is also discussed.

A remarkable property of three dimensional gauge theories coupled to matter is the dynamical generation of the Chern-Simons (CS) topological term[1] through fluctuations of fermionic fields[2, 3, 4]. Non-Abelian CS theories are relevant for their intrinsic topological structure, whereas the interest in the Abelian case stems from the fact that they lead to fractional statistics in 2+1 dimensions. Thus, it is not surprising that such an interaction turns out to be important both in the context of topological field theories and in condensed matter physics, e.g. in phenomena such as the fractional quantum Hall effect and high T_c superconductivity.

It is well known that the classical non-Abelian CS action S_{CS} is not invariant under a homotopically nontrivial gauge transformation which carries non-vanishing winding number. Requiring $\exp(iS_{CS})$ to be gauge invariant leads to the discretization of the CS coefficient[5] at the classical and quantum levels. On the other hand, for an Abelian CS theory, the CS coefficient remains arbitrary due to the absence of a similar topological structure. In what concerns the effective gauge field action $S_{eff}[A]$ (obtained by integrating out the fermionic degrees of freedom), it contains a gauge non-invariant piece which is however exactly cancelled by the parity-violating term arising in any gauge invariant regularization of the fermionic determinant[3].

At finite temperature (or equivalently, when the time component is compactified into a circle) the situation seems to be less understood. Perturbative calculations reveal effective Chern-Simons actions with CS coefficients which are smooth functions of the temperature[6]-[10]. On the contrary, in ref.[11] it was argued that the CS coefficient should be independent of the temperature. This question has been recently addressed in refs.[12, 13], where the authors concluded, on gauge invariance grounds, that the coefficient of the CS term at finite temperature cannot be smoothly renormalized and that it is at most an integer-valued function of the temperature. Nevertheless, calculations of the latter coefficient for an effective action of a massive fermion system in 0+1 dimensions[14] and in the 2+1 Abelian and non-Abelian cases[15, 16], have shown that the complete effective action can be made gauge invariant. Therefore, although the perturbative result yielding a smooth dependence of the CS coefficient on the temperature is correct, any perturbative order is not sufficient to preserve the gauge invariance.

In this letter we shall follow the approach originally used in 2+1 *QED* at zero temperature[3] and compute the parity-violating contribution to the fermionic current at finite temperature for the special case of gauge fields which produce a constant field strength tensor $F_{\mu\nu}$. The present calculation

confirms the fact that gauge invariance of the finite-T Chern-Simons effective action holds for an arbitrary winding number, provided the induced (mass and temperature-dependent) parity breaking contribution as well as the (temperature-independent) parity anomalous contribution are taken into account[15, 16]. In the first order of perturbation theory, the usual perturbative expression for the CS term at finite temperature is also reproduced.

We start by defining the three-dimensional (Euclidean) effective action $S_{eff}[A, m]$ for a massive fermionic field $\psi(\tau, x)$ in a gauge background field $A_\mu(\tau, x)$, with the time direction compactified into the interval $0 \le \tau \le \beta = 1/T$,

$$e^{-S_{eff}[A,m]} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left(- \int_0^\beta d\tau \int d^2x \; \bar{\psi}(\not{\partial} + ie \not{A} + m)\psi \right) \; , \tag{1}$$

where T is the temperature and the Euclidean Dirac matrices are taken in the representation $\gamma_0 = \sigma_3, \gamma_1 = \sigma_1, \gamma_2 = \sigma_2$; σ_μ are the Pauli matrices. The functional integral in Eq.(1) must be evaluated using periodic (antiperiodic) in τ boundary conditions for the gauge (fermionic) fields, i.e.

$$A_\mu(\beta, \mathbf{x}) = A_\mu(0, \mathbf{x}) \; , \tag{2}$$

$$\psi(\beta, \mathbf{x}) = -\psi(0, \mathbf{x}) \; . \tag{3}$$

The allowed non-trivial gauge transformations at finite temperature are those which preserve the above conditions, namely,

$$\begin{aligned} \psi(\tau, \mathbf{x}) &\rightarrow e^{-ie\vartheta(\tau,\mathbf{x})}\psi(\tau, \mathbf{x}) \; , \\ \bar{\psi}(\tau, \mathbf{x}) &\rightarrow e^{ie\vartheta(\tau,\mathbf{x})}\bar{\psi}(\tau, \mathbf{x}) \; , \\ A_\mu(\tau, \mathbf{x}) &\rightarrow A_\mu(\tau, \mathbf{x}) + \partial_\mu\vartheta(\tau, \mathbf{x}) \; , \end{aligned} \tag{4}$$

with

$$\vartheta(\beta, \mathbf{x}) = \vartheta(0, \mathbf{x}) + \frac{2\pi}{e} n \; , \tag{5}$$

where the integer number n characterizes the homotopy class of the gauge transformation.

In what follows we restrict ourselves to gauge field configurations which induce a constant field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and with trivial winding number $n = 0$. We also assume the component A_0 to be independent

of $\mathbf{x}$. Then, the remaining τ dependence in A_0 can always be removed by a redefinition of the fermion fields and the proper choice of a function $\vartheta(\tau)$. Indeed, from the equation $\partial_\tau \vartheta(\tau) = -A_0(\tau) + \bar{A}_0$, with $\bar{A}_0$ a constant, and the condition $\vartheta(\beta) = \vartheta(0)$, we obtain

$$\vartheta(\tau) = -\int_0^\tau A_0(\tau) d\tau + \bar{A}_0 \tau \ , \quad \bar{A}_0 = \frac{1}{\beta} \int_0^\beta A_0(\tau) d\tau.$$

Under the above assumptions, the gauge field A_μ can be written as

$$A_\mu = (A_0, \frac{1}{2} F_{ij} x_j) \ ; \ A_0 \equiv \bar{A}_0; \quad i,j = 1,2 \ . \tag{6}$$

We note that this choice corresponds to a vanishing electric field ($F_{0i} = 0$) and a constant magnetic field. This gauge configuration will allow us to compute exactly the Chern-Simons topological current at finite temperature, following an approach similar to the zero-temperature case[3].

The ground-state current in the presence of the background field (6) is defined as

$$\langle J^\mu \rangle = \frac{\delta S_{eff}[A,m]}{\delta A_\mu} \ , \tag{7}$$

where the total effective action is given, according to Eq.(1), by

$$S_{eff}[A,m] = -\mathrm{Tr} \ln(\partial\!\!\!/ + ie A\!\!\!/ + m) \ . \tag{8}$$

Notice that since the only parity-odd term in the Euclidean effective action is the mass term, we can obtain the CS current through the combination

$$\langle J^\mu_{CS} \rangle = \frac{1}{2} \left(\frac{\delta S_{eff}[A,m]}{\delta A_\mu} - \frac{\delta S_{eff}[A,-m]}{\delta A_\mu} \right) \ . \tag{9}$$

The (gauge-invariant) background current (7) can be written in terms of the Green function for the Dirac operator[17],

$$\langle J^\mu \rangle = e \mathrm{Tr} \left[\gamma^\mu \exp\left(-ie \int_x^{x'} A(\xi) \cdot d\xi \right) G(x,x') \right] \Bigg|_{x \to x'} \ , \tag{10}$$

where

$$G(x,x') = \langle x | G | x' \rangle = \left\langle x \left| (\Pi\!\!\!/ + im)^{-1} \right| x' \right\rangle \ , \tag{11}$$

$x^\mu \equiv (\tau, \mathbf{x})$,$\Pi_\mu = p_\mu - eA_\mu$ and $[\Pi_\mu, \Pi_\nu] = ieF_{\mu\nu}$. As customary at finite temperature, calculations are carried out by restricting the values of the Euclidean time τ to the interval $0 \leq \tau \leq \beta = 1/T$. We have then

$$G(x, x') = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} G \ e^{ip(x-x')}, \tag{12}$$

with $p_0 = \omega_n = (2n+1)\pi/\beta$ - the Matsubara frequencies; $px = \omega_n \tau + \mathbf{px}$.

To calculate explicitly $\langle J^\mu_{CS} \rangle$, let us write the operator G in the equivalent form[17]

$$G = \frac{1}{\not\Pi + im} = (\not\Pi - im) \int_0^\infty ds \ e^{-s\left(\not\Pi^2 + m^2\right)}. \tag{13}$$

According to the definition given in Eq.(9), it is easy to see that the only contribution to the CS current comes from the term proportional to im in Eq.(13). Thus, we shall consider the simplified function

$$G_{CS} = -\frac{im}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \int_0^\infty ds \ e^{-s\left(\not\Pi^2 + m^2\right)} e^{ip(x-x')} \Big|_{x \to x'} . \tag{14}$$

Using the relation $\not\Pi^2 = \Pi^2 + \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu}, \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, and the fact that $[\Pi_0, \Pi_i] = 0$ for the gauge configuration (6), it is easy to factorize the τ dependence in Eq.(14) to obtain

$$G_{CS} = -\frac{im}{\beta} \sum_{n=-\infty}^{\infty} \int_0^\infty ds \ e^{-s\left[(\omega_n - eA_0)^2 + \frac{e}{2}\sigma \cdot F + m^2\right]} \left\langle \mathbf{x} \left| e^{-s\mathbf{\Pi}^2} \right| \mathbf{x}' \right\rangle \Big|_{\mathbf{x} \to \mathbf{x}'} . \tag{15}$$

The matrix element appearing in the last equation can be evaluated using the method developed by Schwinger[17] to perform similar calculations in four-dimensional *QED*. We define the evolution operator $U(s) = e^{-s\mathcal{H}}, \mathcal{H} = \mathbf{\Pi}^2$, which describes the development in the proper (imaginary) time s of a system with Hamiltonian $\mathcal{H}$. Then

$$\left\langle \mathbf{x} \left| e^{-s\mathbf{\Pi}^2} \right| \mathbf{x}' \right\rangle = \langle \mathbf{x} | U(s) | \mathbf{x}' \rangle = \langle \mathbf{x}, s \mid \mathbf{x}', 0 \rangle \ , \tag{16}$$

with the boundary condition $\langle \mathbf{x}, s \mid \mathbf{x}', 0 \rangle |_{s \to 0} = \delta(\mathbf{x} - \mathbf{x}')$.

Solving the associated dynamical problem

$$\frac{dx_i}{ds} = [H, x_i] = -2i\Pi_i \ , \ \frac{d\Pi_i}{ds} = [H, \Pi_i] = -2ieF_{ij}\Pi_j \ , \tag{17}$$

finally we obtain

$$\langle \mathbf{x}, s \mid \mathbf{x}', 0\rangle|_{\mathbf{x}\to\mathbf{x}'} = \frac{1}{4\pi}\frac{e|\tilde{F}|}{\sinh(es|\tilde{F}|)} \ , \tag{18}$$

where $\tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\alpha\beta}F_{\alpha\beta}$ is the dual to $F_{\mu\nu}$ and $|\tilde{F}| = \sqrt{B^2 - E^2}$.

In evaluating the Chern-Simons current (cf. Eq.(10)) we also need the following trace formula:

$$\mathrm{Tr}\left[\gamma^\mu e^{-\frac{es}{2}\sigma\cdot F}\right] = -2\frac{\tilde{F}^\mu}{|\tilde{F}|}\sinh(es|\tilde{F}|). \tag{19}$$

From Eqs.(10),(15),(18) and (19) we find

$$\begin{aligned} \langle J^\mu_{CS}\rangle &= \frac{ime^2\tilde{F}^\mu}{2\pi\beta}\sum_{n=-\infty}^{\infty}\int_0^\infty ds\ e^{-s\left[(\omega_n - eA_0)^2 + m^2\right]} \\ &= \frac{ime^2\tilde{F}^\mu}{2\pi\beta}\sum_{n=-\infty}^{\infty}\frac{1}{(\omega_n - eA_0)^2 + m^2} \ . \end{aligned} \tag{20}$$

The last sum in the above equation is easily evaluated and we arrive at our final result:

$$\langle J^\mu_{CS}\rangle = \frac{ie^2}{4\pi}\frac{m}{|m|}\tilde{F}^\mu\frac{\sinh(\beta|m|)}{\cosh(\beta|m|) + \cos(e\beta A_0)} \ . \tag{21}$$

First notice that in the limit $T \to 0$ we reproduce the standard zero temperature result[3], namely,

$$\langle J^\mu_{CS}\rangle = \frac{ie^2}{4\pi}\frac{m}{|m|}\tilde{F}^\mu. \tag{22}$$

We can now functionally integrate over the field A_μ in Eq.(20) to obtain the Chern-Simons effective action

$$S_{CS}[A] = \frac{ie}{2\pi}\frac{m}{|m|}\arctan\left[\tanh\left(\frac{\beta|m|}{2}\right)\tan\left(\frac{e\beta A_0}{2}\right)\right]\int d^2x\epsilon_{ij}\partial_i A_j \ . \tag{23}$$

This result coincides with the one recently obtained in refs.[15, 16] by calculating the fermion determinant. In particular, when $T \to 0$ it gives the

standard zero-temperature result

$$S_{CS}[A] = \frac{ie^2}{8\pi}\frac{m}{|m|}\int d^3x \tilde{F}_\mu A_\mu = \pm 2\pi W[A], \tag{24}$$

where the $\pm$ sign depends on the sign of m and $W[A]$ is the Chern-Simons secondary characteristic class number[5],

$$W[A] = \frac{ie^2}{16\pi^2}\int d^3x \tilde{F}_\mu A_\mu \ . \tag{25}$$

We note that under non-trivial gauge transformations with winding number $n \neq 0$, the argument of $\tan(e\beta A_0/2)$ in Eq.(23) is shifted by $n\pi$ and so does the value of the arctan function, once the branch used in its definition is also properly shifted. Thus, a situation similar to the $T = 0$ case is occurring, namely, the gauge non-invariance of the Chern-Simons effective action is compensated by the parity anomalous contribution[3], which arises in any gauge-invariant regularization scheme and amounts to $\pm 2\pi W[A]$.

Eq.(23) also reproduces in the first order of perturbation (in e) theory, the usual perturbative expression:

$$S_{CS}[A, m] = \pm 2\pi \tanh\left(\frac{\beta|m|}{2}\right) W[A] \ . \tag{26}$$

This perturbative result shows however an apparent breaking of gauge invariance: the gauge non-invariance present in the effective action (26) is no longer cancelled by the (temperature-independent) parity anomaly. From this we can conclude that any perturbative order is not sufficient to preserve gauge invariance and thus, a full (non-perturbative) answer is necessary in order to discuss gauge invariance at finite temperature.

Finally, let us briefly comment on the non-Abelian case. Similar calculations can be performed for $SU(N)$ non-Abelian gauge-field configurations which induce a constant field strength tensor $F^a_{\mu\nu}$ (a is the colour index). As is well-known, there are only two types of such fields[3]. The first one is an Abelian-type field, which following arguments similar to the ones presented here(see Eq.(6)) can be written in the form $A^a_\mu = (\eta^a A_0, \frac{1}{2}\eta^a F_{ij}x_j)$, where η^a is a constant unit vector in colour space. The second one is a genuine non-Abelian constant gauge field, $A^a_\mu = const$, with $[A_\mu, A_\nu] \neq 0$, $A_\mu \equiv A^a_\mu T_a$ and T_a the generators of the Lie algebra. For such fields, the gauge invariance of the effective Chern-Simons action under non-trivial gauge transformations is a more subtle[16] question which deserves further study.

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